

# Polyelectrolytes in the presence of multivalent ions: gelation versus segregation

A. V. Ermoshkin, M. Olvera de la Cruz  
*Department of Materials Science and Engineering,  
 Northwestern University, Evanston, IL, USA 60208-3108*  
 (February 1, 2008)

We analyze solutions of strongly charged chains bridged by linkers such as multivalent ions. The gelation induced by the strong short range electrostatic attractions is dramatically suppressed by the long range electrostatic correlations due to the charge along the uncrosslinked monomers and ions. A modified Debye-Hückel approach of crosslinked clusters of charged chains is used to determined the mean field gelation transition self-consistently. Highly dilute polyelectrolyte solutions tend to segregate macroscopically. Semidilute solutions can form gels if the Bjerrum length  $l_B$  and the distance between neighboring charged monomers along the chain  $b$  are both greater than the ion size  $a$ .

Linear polyelectrolytes are ubiquitous in biology given that nucleic acids and most proteins are charged. Moreover, they have important technological applications as gelling and drag reduction agents. Long linear polyelectrolytes are typically water soluble in low ionic strength monovalent salt solutions due to a net repulsion between the charged monomers. Linking agents such as multivalent ions of valence  $z > 1$ , however, modify the stability of polyelectrolytes aqueous solutions [1–5]. The stability of biopolymers in various ionic media has important implications in biotechnological processes [6].

The solubility of linear polyelectrolyte in multivalent salts has been extensively studied in dilute solutions of continuously charged flexible chain [7] and related systems [8]. Only few studies, however, have discussed the stability of the chains in semidilute solutions [2,3]. In particular, the strong binding between small (dehydrated) multivalent metallic ions and charged ( $O^-$ ) groups in different chains opens the possibility of a gelation transition in semidilute solutions [9,10]. The competition of gelation and phase separation has been recently studied in neutral chains with linkers [11,12]. The charge along the chains, however, will strongly modify the phase diagram. Here we describe the gelation by multivalent ions including long and short range correlation self-consistently in the analysis.

Our model treats the strong short range correlations driven by the multivalent cations explicitly using bridges or crosslinks that change the structure function of the system and therefore the long range electrostatic correlations. The long range correlations are accounted via a modified mean field approach known as the Random Phase Approximation (RPA) on the linked system. RPA

describes properly long range correlations in polyelectrolyte solutions without short range correlations [13]. Therefore, by using as a reference state of RPA a strongly correlated system of chains linked by multivalent ions, we can in principle determine self-consistently the gelation transition in the system of charged chains. We consider only flexible charged chains to avoid difficulties found in the gel structures of charged rigid rods [14] due to the strong orientational dependent electrostatic interactions [15,16].

Consider a polymer solutions where every chain monomer carries a negative charge. The concentration of monomers is  $\rho_m$ , and of monovalent counterions is  $\rho_m^c = \rho_m$ , and we assume for simplicity that all of them are completely dissociated from the chains. We denote the concentrations of multivalent ions and their counterions as  $\rho_i$  and  $\rho_i^c$ , respectively ( $\rho_i^c = z\rho_i$ ). Let us assume that  $z = 2$ . The interaction between the multivalent ions and the chains, in principle, leads to the formation of monocomplexes ( $mc$ ) and dicomplexes ( $dc$ ). A  $mc$  is formed when an ion binds onto one negative site of the polyelectrolyte. Formation of a  $dc$  brings two negative sites together. The competition between the formation of  $mc$  and  $dc$  is described by the process  $2mc \longleftrightarrow dc + \text{free ion}$ . The equilibrium constant  $p$  of this process can be estimated as [3]  $p = K_2 \exp(\kappa l_B) / K_1^2$ , where  $l_B = e^2 / \epsilon k_B T$  is the Bjerrum length and the inverse screening length  $\kappa$  is given by  $\kappa^2 = 4\pi l_B \sum z_\alpha^2 \rho_\alpha$  with the sum taken over all dissociated charges.  $K_1$  and  $K_2$  are the statistical weights of  $mc$  and  $dc$  respectively. Assuming that all ions and the monomers of the chains have the same size  $a$  we can estimate these weights following Fuoss [17]

$$K_1 = \frac{4\pi}{3} a^3 \exp(l_B/a) \quad (1)$$

$$K_2 = \frac{1}{2} \left( \frac{4\pi}{3} a^3 \right)^2 \exp(3l_B/a) \quad (2)$$

As shown in Ref. [3] dicomplexation is favorable if  $p > 1/12$ . Since in our work we consider strongly charged chains regimes with  $p \sim 10$ , we neglect the possibility of forming monocomplexes.

Dicomplexation processes may have two different scenarios. If the two sites that form a dicomplex belong to the same chain then this chain collapses onto itself. If these sites, however, are from different chains then a crosslink is formed. The monomolecular collapse is ob-

served when the multivalent ions nearly neutralize the chains at [7]

$$\rho_i^{\text{coll}} = \frac{\rho_m}{z} \quad (3)$$

Here we consider the formation of branched structures by crosslinks at  $\rho_i < \rho_i^{\text{coll}}$ .

We write the free energy of the system in the form

$$F = F^* + F_{\text{ref}} + F_{\text{el}} \quad (4)$$

$F^*$  accounts for excluded volume interactions and can include small effective short range monomer attractions ( $\chi \leq 2$ ) to account for slightly hydrophobic chain backbones. It can be approximated as

$$\frac{F^* a^3}{TV} = (1 - \phi) \log(1 - \phi) - \chi \phi_m^2 \quad (5)$$

Here  $T$  is the temperature in units of  $k_B$ ,  $V$  is the volume,  $\phi_m = a^3 \rho_m$ , and  $\phi$  is the total volume fraction of all components  $\phi = a^3(\rho_m + \rho_m^c + \rho_i + \rho_i^c)$ .  $F_{\text{ref}}$  corresponds to free energy of the system without any long range electrostatic interactions, the reference system,

$$\frac{F_{\text{ref}}}{TV} = \sum_{\{C\}} \rho_C \ln \frac{\rho_C s_C}{e w_C} \quad (6)$$

The sum is taken over all possible structures  $C$  of clusters formed due to crosslinking ( $\{C\}$  also includes trivial clusters such as free ions and counterions), and  $\rho_C$ ,  $w_C$  and  $s_C$  represent concentration, statistical weight and symmetry index of a cluster  $C$ , respectively.

The last term  $F_{\text{el}}$  in Eq. (4) takes into account all long range electrostatic effects and can be written in the form [18]

$$\frac{F_{\text{el}}}{TV} = \frac{1}{2(2\pi)^3} \int \left[ \ln \frac{\det \|U_{\mathbf{k}}^{ij} + (g^{-1})_{\mathbf{k}}^{ij}\|}{\det \|(g^{-1})_{\mathbf{k}}^{ij}\|} - \sum_i \rho_i U_{\mathbf{k}}^{ii} \right] d\mathbf{k} \quad (7)$$

where the matrix of correlation functions  $\mathbf{g}_{\mathbf{k}} = \|g_{\mathbf{k}}^{ij}\|$  has the following form

$$g_{\mathbf{k}}^{ij} = \sum_C \rho_C g(C)_{\mathbf{k}}^{ij} \quad (8)$$

Here  $g(C)_{\mathbf{k}}^{ij}$  is the scattering factor of a cluster characterized by a structure  $C$

$$g(C)_{\mathbf{k}}^{ij} = \sum_{m,n} \left\langle e^{i\mathbf{k}(\mathbf{r}_i^m - \mathbf{r}_j^n)} \right\rangle_C \quad (9)$$

with  $m$  and  $n$  running over all  $i$ -type and  $j$ -type units in  $C$  cluster respectively.

We calculate free energy of the reference system assuming that all complex clusters have only tree-like architectures [3,19,20]

$$F_{\text{ref}} = F_{\text{ref}}^{\text{id}} + F_{\text{ref}}^{\text{comb}} + F_{\text{ref}}^{\text{cross}} \quad (10)$$

$F_{\text{ref}}^{\text{id}}$  is associated with all translational entropies,

$$\frac{F_{\text{ref}}^{\text{id}}}{TV} = \frac{\rho_m}{N} \ln \rho_m + \rho_m^c \ln \rho_m^c + \rho_i \ln \rho_i + \rho_i^c \ln \rho_i^c \quad (11)$$

$F_{\text{ref}}^{\text{comb}}$  is the term obtained from the number of possibilities to choose the monomers and the ions that participate in formation of crosslinks,

$$\frac{F_{\text{ref}}^{\text{comb}}}{TV} = \rho_m (\Gamma_m \ln \Gamma_m + (1 - \Gamma_m) \ln(1 - \Gamma_m)) + \rho_i (\Gamma_i \ln \Gamma_i + (1 - \Gamma_i) \ln(1 - \Gamma_i)) \quad (12)$$

where  $\Gamma_m$  and  $\Gamma_i$  are the fractions of the monomers and the ions respectively that belong to the crosslinks. Because we neglect the formation of monocomplexes these fractions are related by  $\rho_m \Gamma_m = z \rho_i \Gamma_i$ . The last term in Eq. (10) corresponds to the crosslinking free energy,

$$\frac{F_{\text{ref}}^{\text{cross}}}{TV} = -\rho_m \Gamma_m \ln \frac{\rho_m \Gamma_m K_2^{1/2}}{e} \quad (13)$$

We use Eqs. (10)-(13) for the reference free energy to determine the instability of the system due to the formation of the infinitely large network or gel. The gelation line is identified by the divergence of the cluster's weight average  $N_w$ . We note that  $N_w \rightarrow \infty$  corresponds to a singularity of the second derivatives of the reference free energy [19]

$$\det \left\| \frac{\partial^2 F_{\text{ref}}}{\partial \rho_\alpha \partial \rho_\beta} \right\| = 0 \quad (14)$$

Here the set  $\{\rho_\alpha\}$  includes  $\rho_i$ ,  $\rho_m$  and  $\rho_m^{\text{cross}} = \rho_m \Gamma_m$ . Representing  $F_{\text{ref}}$  only as a function of these three concentrations and evaluating the determinant (14) we get the critical value  $\Gamma_m^*$  that corresponds to the formation of the network

$$\Gamma_m^* = \frac{1}{(z-1)(N-1)} \quad (15)$$

The last term of the free energy  $F_{\text{el}}$  given by Eq. (7) is calculated as follows. First we evaluate the  $5 \times 5$  matrix of correlation functions  $\mathbf{g}_{\mathbf{k}}$  which has the following components:  $g_{11} = \rho_m^c$  is the concentration of counterions dissociated from the polymer chains,  $g_{22} = \rho_i(1 - \Gamma_i)$  is the concentration of free multivalent ions,  $g_{33} = \rho_i^c$  is the concentration of counterions dissociated from the multivalent ions, and the correlation functions between the connected units are denoted by  $g_{44}$ ,  $g_{45} = g_{54}$  and  $g_{55}$ , where 4 and 5 stand for non-crosslinked monomers and crosslinked aggregates respectively. In the present work we neglect dipole-dipole and dipole-charge interactions, therefore,  $g_{45}$  and  $g_{55}$  have no effect on the free energy of the system. We calculate the monomer-monomer

correlations  $g_{44}$  using the diagrammatic technique described in Ref. [21]. By introducing the concentration  $t$  of all branching structures that could be attached to a chain monomer, and the concentration  $\mathcal{T}$  of all non-associated monomers, we obtain the total concentration of monomers  $\rho_m = t\mathcal{T}$ , and the fraction of associated monomers  $\Gamma_m = 1 - 1/t$ . By finding the correlation function between two monomers that belong to the same chain  $\Sigma(k) = (\mathcal{T}/t)(g(k) - 1)$  and taking into account all possible arrangements of linear chains into tree-like structures we get

$$g_{44}(k) = \rho_m(1 - \Gamma_m) \frac{1 + (1 - z\Gamma_m)(g(k) - 1)}{1 - \Gamma_m(z - 1)(g(k) - 1)} \quad (16)$$

where  $g(k)$  is the structure factor of a chain defined as

$$g(k) = \frac{1}{N} \sum_{i,j} \langle e^{-i\mathbf{k}(\mathbf{r}_i - \mathbf{r}_j)} \rangle \quad (17)$$

Strongly charged chains in semidilute solutions are stretched at length scales of few monomers  $n$  due to the locally unscreened electrostatic repulsions. We assume they behave like rods on short length scales, although simulations reveal slightly less stretched local conformations [22]. On larger scales the chains obey Gaussian statistics with  $N/n$  segments per chain. For  $g(k)$  we use the approximated expression

$$g(k) - 1 = \frac{n - 1}{1 + nkb} + \frac{N - n}{1 + Nnk^2b^2/12} \quad (18)$$

where  $b$  is the average distance between the neighbor monomers along the chain, which gives the right limits and it is reasonably accurate. Though in ref. [22] it is argued that  $n$  is nearly  $\phi_m$  independent, we estimated it here as [23]

$$n = \frac{1}{(\phi_m b^3)^{1/2}} \quad (19)$$

to obtain a bound on the gelation in the largest possible perturbed chain conformation. We also calculate here the gelation for the least perturbed chain conformations, Gaussian chains, by putting  $n = 1$  in Eq. (18); ideal chain statistics are observed in charged chains in certain hydrophobic backbone regimes [24].

When the fraction of associated monomers  $\Gamma_m$  reaches the critical value  $\Gamma_m^*$  given by Eq. (15) then the correlation function  $g_{44}(k)$  diverges at  $k = 0$ . This divergence represents correlations on the infinite scale and, as discussed above, corresponds to the network formation.

To obtain the long range electrostatic contribution to the free energy given by Eq. (7) we introduce the following interaction potential between the charges [25]

$$U^{ij}(r) = z_i z_j l_B \frac{1 - e^{-r/a}}{r} \quad (20)$$

which allows us to account for short range repulsions due to the hard-sphere nature of the charges.

Taking the Fourier transform of  $U^{ij}(r)$  we write the electrostatic free energy (7) in the form

$$\frac{F_{el}}{TV} = \frac{1}{4\pi^2} \int \left[ \ln(1 + U(k)(g_{44}(k) + \kappa^2)) - U(k)(\rho_m(1 - \Gamma_m) + \kappa^2) \right] k^2 dk \quad (21)$$

where  $\kappa^2 = \rho_m + z\rho_i + z^2\rho_i(1 - \Gamma_i)$  and

$$U(k) = \frac{4\pi l_B}{k^2(1 + a^2 k^2)} \quad (22)$$

We obtain the fraction of crosslinked monomers in the system  $\Gamma_m^{\min}$ , for given  $\rho_m$ ,  $\rho_i$  and  $l_B/b$  by minimizing the total free energy with respect to  $\Gamma_m$ . Because the correlation function  $g_{44}(0)$  diverges at  $\Gamma_m = \Gamma_m^*$  we find a minimum of the free energy in the interval  $\Gamma_m \in [0, \Gamma_m^*]$ . If the minimum is reached inside this interval then the system contains only finite size clusters. If the minimum is reached at  $\Gamma_m^{\min} = \Gamma_m^*$  then an infinitely large network is formed in the system. By numerically solving

$$\Gamma_m^{\min}(\rho_i^*, \rho_m, K(\xi)) = \Gamma_m^* \quad (23)$$

we obtain the gelation line  $\rho_i^*(\rho_m)$ . For a given  $\rho_m$ , if  $\rho_i < \rho_i^*$  the system contains only finite clusters, if  $\rho_i \geq \rho_i^*$  an infinitely large network or gel is formed. The results are shown in Figure 1. Above the straight line which corresponds to Eq. (3) the chains are collapsed. The dash line labelled as 1 is the gelation line when the long range electrostatic interactions are not included (i.e., setting  $F_{el} = 0$  in Eq. (21)), and it is independent of the chain structure function in the mean field model. Line 2 is the gelation transition obtained for locally stretched chains (the structure factor given by Eq. (18)), and line 3 is the gelation obtained for Gaussian chain statistics at all length scales ( $n = 1$ ). The gelation transition changes by orders of magnitude and depends on the chain conformation when long range electrostatic interactions are included.

In Figure 2 we show the gelation lines obtained for different values of  $l_B/b$  and  $b/a$ . Notice that the ion size  $a$  in  $K_2$  in Eq. (2) is a parameter different than the average distance between the neighboring monomers along the chain  $b$ . It can be varied to account for the unknown size of the ions in the crosslinks and the unknown dielectric constant of the local medium around the links. For  $a = b$  the gelation occurs at rather high  $\rho_i$  concentrations. This may explain why in solutions of strongly charged polyelectrolytes, such as DNA and Polystyrene-Sulphonate, where both metallic and organic multivalent ions are hydrated around the chains ( $a > b$ ),  $z \geq 3$  are required to observe precipitation, and the gelation is not observed at  $\rho_i^* \simeq \rho_m/(Nz(z - 1))$  even in semidilute solutions but at

much larger values of  $\rho_i$ . Instead in polyelectrolytes with monomers with ionizable OH groups such as acrylate groups, since metallic ions are dehydrated when forming dicomplexes, divalent metallic ions do precipitate the chains, and gels are formed in semidilute solutions [10].

We investigated the stability of the system to macroscopic phase separation, and found no instabilities induced by electrostatics below the gelation lines for the range of parameters used here ( $l_B/b \sim 2$ ,  $\chi \leq 2$ ). Thus we conclude that there is no segregation transition that competes with the gelation in the regimes studied here.

In conclusion, we have determined the effect of long range electrostatic interactions on gelation of polyelectrolyte chains induced by multivalent ions. The approach can be used to describe gelation of charged chains bridged by other type of linkers through modification of  $K_2$ . We could also account for variable weights of different types of crosslink functionalities  $m$  by allowing a  $K_m$  dependence. It is of future interest to estimate the effect of cyclization of the chains in the gelation theory of polyelectrolytes.

We acknowledge the financial support of the NIH grant number GM62109-02, the Institute for Bioengineering and Nanosciences in Advanced Medicine (IBNAM) at Northwestern University and the NSF grant number EEC-0118025.

- 
- [1] J. Widom, and R. L. Baldwin, J. Mol. Biol. **144**, 431 (1980).
  - [2] M. Olvera de la Cruz, L. Belloni, M. Delsanti, J. P. Dalbiez, O. Spalla, M. Drifford, J. Chem. Phys. **103**, 5871 (1995).
  - [3] J. Wittmer, A. Johner, J. F. Joanny, J. Phys. II **5**, 635 (1995).
  - [4] E. Raspaud, M. Olvera de la Cruz, J.L. Sikorav, F. Livolant, Biophys. Journal **74**, 381 (1998).
  - [5] K. Yoshikawa, M. Takahashi, V.V. Vasilevskaya, A. R. Khokhlov, Phys. Rev. Lett. **76**, 3029 (1996).
  - [6] D. Jary, J.-L. Sikorav, Biochemistry **38**, 3223 (1999).
  - [7] F. J. Solis, M. Olvera de la Cruz, Euro. Phys. J. E **4**, 143 (2001) and references therein.
  - [8] I. Rouzina and V. Bloomfield, J. Phys. Chem. **100** 9977, (1996). For an excellent review see Y. Levin, <http://xxx.lanl.gov/abs/cond-mat/0207086>.
  - [9] J. F. Joanny, Polymer **21**, 71 (1980).
  - [10] I. Sabbagh, M. Delsanti, Eur. Phys. J. E **1**, 75 (2000).
  - [11] S. K. Kumar and J. F. Douglas, Phys. Rev. Lett. **87**, 188301 (2001)
  - [12] I. Ya. Erukhimovich and A. V. Ermoshkin, J. Chem. Phys. **116**, 368 (2002).
  - [13] N. Bagatella-Flores and P. González-Mozuelos, J. Chem. Phys. (in press).
  - [14] I. Borukhov, R.F. Bruinsma, Phys. Rev. Lett. **87**, 158101 (2001).

- [15] M. Muthukumar, G. Carri, J. Chem. Phys. **111**, 1765 (1999).
- [16] I. Borukhov, K. C. Lee, R. F. Bruinsma, W. M. Gelbart, A. J. Liu, and M. J. Stevens, J. Chem. Phys. **117**, 462 (2002).
- [17] R. M. Fuoss, J. Amer. Chem. Soc. **80**, 5059 (1958).
- [18] V. Yu. Borue and I. Ya. Erukhimovich, Macromolecules **21**, 3240 (1988).
- [19] I. Ya. Erukhimovich, JETP **81**, 553 (1995).
- [20] A. N. Semenov and M. Rubinstein, Macromolecules **31**, 1373 (1998).
- [21] A. V. Ermoshkin and I. Ya. Erukhimovich, J. Chem. Phys. **110**, 1781 (1999).
- [22] M. J. Stevens, K. Kremer J. Chem. Phys. **103**, 1669 (1995).
- [23] A. V. Dobrynin, M. Rubinstein, Macromolecules **32**, 915 (1999). H. Schiessel, Macromolecules **32**, 5673 (1999).
- [24] U. Micka, C. Holm, K. Kremer Langmuir **15**, 4033 (1999).
- [25] This potential reproduces phase diagrams generated by approaches that include hard core potentials explicitly (A. V. Ermoshkin, M. Olvera de la Cruz, to be published)

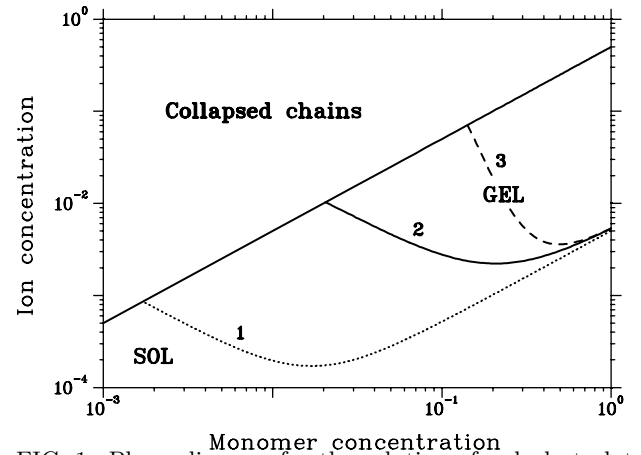


FIG. 1. Phase diagram for the solution of polyelectrolyte chains in the presence of divalent ions. Concentrations are given in  $a^3$  units,  $l_B/b = 2$ ,  $b = a$ ,  $N = 100$ .

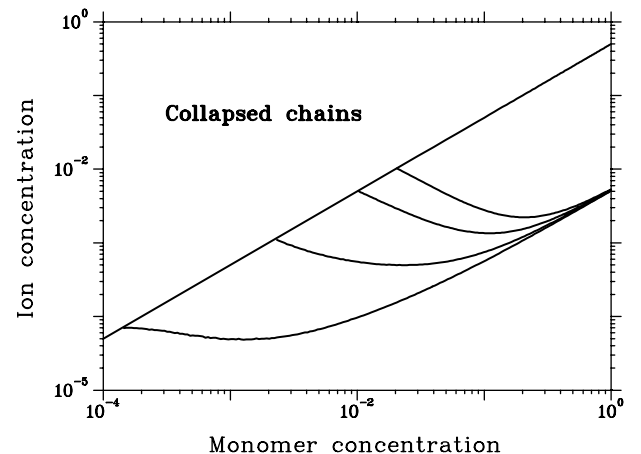


FIG. 2. Gelation lines for different values of parameters  $l_B/b$  and  $b/a$ . From top to bottom  $(l_B/b, b/a) = (2,1)$ ,  $(3,1)$ ,  $(2,1.5)$ ,  $(3,1.5)$ . Concentrations are given in  $a^3$  units,  $N = 100$ .